

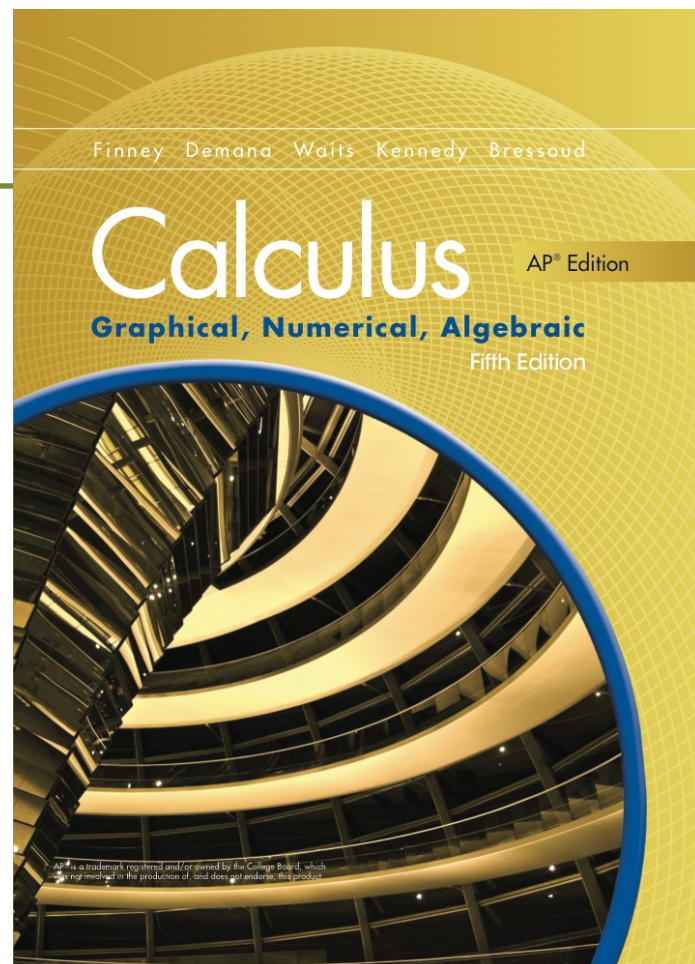
Chapter 4

More Derivatives



Section 4.2

Implicit Differentiation



Quick Review

In Exercises 1– 5, sketch the curve defined by the equation and find two functions y_1 and y_2 whose graphs will combine to give the curve.

1. $x - y^2 = 0$

2. $4x^2 + 9y^2 = 36$

3. $x^2 - 4y^2 = 0$

4. $x^2 + y^2 = 9$

5. $x^2 + y^2 = 2x + 3$

Quick Review

In Exercises 6–8, solve for y' in terms of y and x .

6. $x^2 y' - 2xy = 4x - y$

7. $y' \sin x - x \cos x = xy' + y$

8. $x(y^2 - y') = y'(x^2 - y)$

Quick Review

In Exercises 9 and 10, find an expression for the function using rational powers rather than radicals.

9. $\sqrt{x}(x - \sqrt[3]{x})$

10. $\frac{x + \sqrt[3]{x^2}}{\sqrt{x^3}}$

Quick Review Solutions

In Exercises 1– 5, sketch the curve defined by the equation and find two functions y_1 and y_2 whose graphs will combine to give the curve.

1. $x - y^2 = 0$ $y_1 = \sqrt{x}$, $y_2 = -\sqrt{x}$

2. $4x^2 + 9y^2 = 36$ $y_1 = \frac{2}{3}\sqrt{9 - x^2}$, $y_2 = -\frac{2}{3}\sqrt{9 - x^2}$

3. $x^2 - 4y^2 = 0$ $y_1 = \frac{x}{2}$, $y_2 = -\frac{x}{2}$

4. $x^2 + y^2 = 9$ $y_1 = \sqrt{9 - x^2}$, $y_2 = -\sqrt{9 - x^2}$

5. $x^2 + y^2 = 2x + 3$ $y_1 = \sqrt{2x + 3 - x^2}$, $y_2 = -\sqrt{2x + 3 - x^2}$

Quick Review Solutions

In Exercises 6–8, solve for y' in terms of y and x .

$$6. \quad x^2 y' - 2xy = 4x - y \qquad y' = \frac{4x - y + 2xy}{x^2}$$

$$7. \quad y' \sin x - x \cos x = xy' + y \qquad y' = \frac{y + x \cos x}{\sin x - x}$$

$$8. \quad x(y^2 - y') = y'(x^2 - y) \qquad y' = \frac{xy^2}{x^2 - y + x}$$

Quick Review Solutions

In Exercises 9 and 10, find an expression for the function using rational powers rather than radicals.

$$9. \quad \sqrt{x} \left(x - \sqrt[3]{x} \right) \qquad x^{\frac{3}{2}} - x^{\frac{5}{6}}$$

$$10. \quad \frac{x + \sqrt[3]{x^2}}{\sqrt{x^3}} \qquad x^{\frac{-1}{2}} + x^{\frac{-5}{6}}$$

What you'll learn about

- Implicitly defined functions
- Using the Chain Rule to find derivatives of functions defined implicitly
- Tangent and normal lines to implicitly defined curves
- Finding higher order derivatives of implicitly defined functions
- Extending the Power Rule from integer powers to rational powers

... and why

Implicit differentiation allows us to find derivatives of functions that are not defined or written explicitly as a function of a single variable.

Implicitly Defined Functions

An important problem in Calculus is how to find the slope when the function can't conveniently be solved for y . Instead, y is treated as a differentiable function of x and both sides of the equation are differentiated with respect to x , using the appropriate rules for sums, products, quotients and the Chain rule. Then solve for $\frac{dy}{dx}$ in terms of x and y together to obtain a formula that calculates the slope at any point (x, y) on the graph.

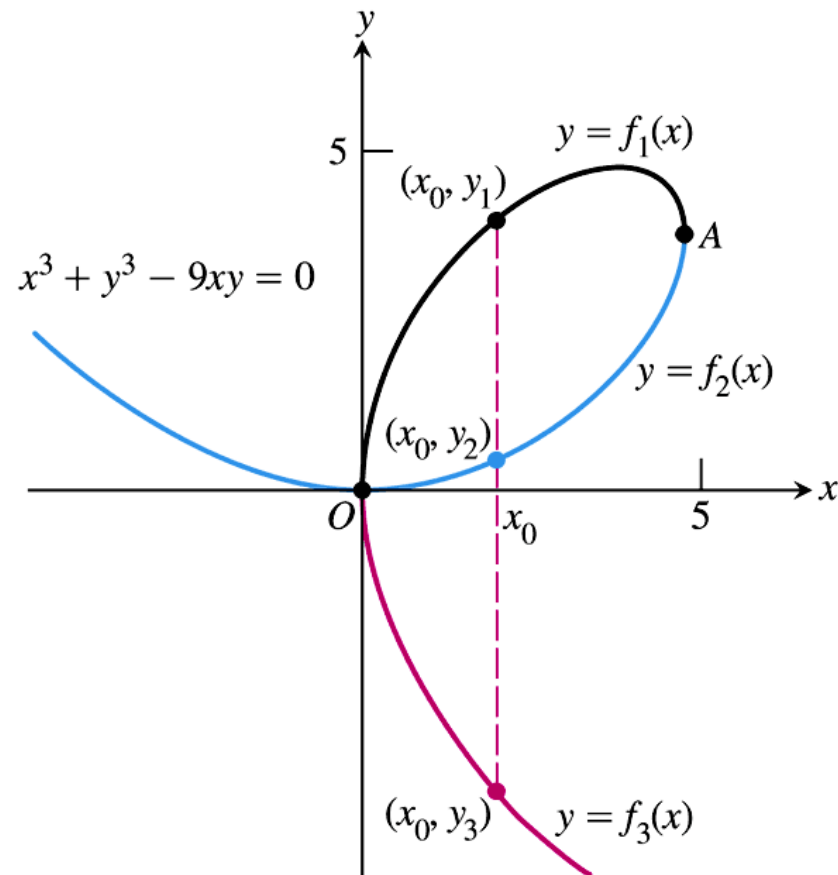
The process by which we find $\frac{dy}{dx}$ is called **implicit differentiation**.

The phrase derives from the fact that the equation

$$x^3 + y^3 - 9xy = 0$$

defines the functions f_1 and f_3 implicitly (i.e., hidden inside the equation), without giving us *explicit* formulas to work with.

Implicitly Defined Functions



Example Implicitly Defined Functions

Find $\frac{dy}{dx}$ if $3y^2 + 2y = 5x$

To find $\frac{dy}{dx}$ differentiate both sides of the equation with respect to x , treating y as a differentiable function of x and applying the Chain Rule.

$$3y^2 + 2y = 5x$$

$$6y \frac{dy}{dx} + 2 \frac{dy}{dx} = 5$$

$$\begin{cases} \frac{d}{dx}(3y^2) = \frac{d}{dy}(3y^2) \frac{dy}{dx} \\ \frac{d}{dx}(2y) = \frac{d}{dy}(2y) \frac{dy}{dx} \end{cases}$$

$$\frac{dy}{dx}(6y + 2) = 5$$

$$\frac{dy}{dx} = \frac{5}{6y + 2}$$

Implicit Differentiation Process

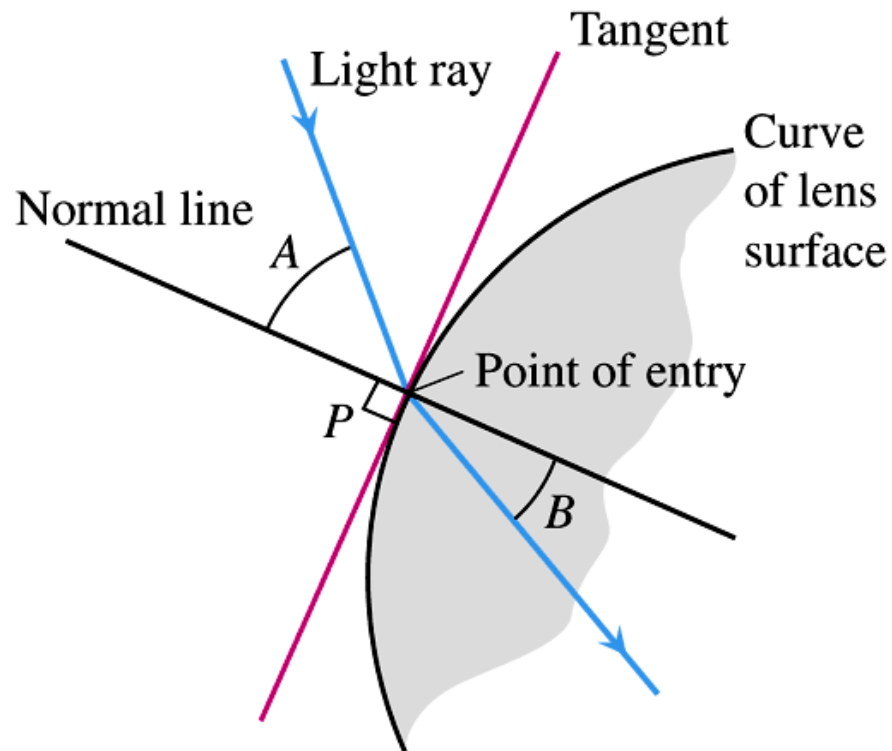
1. Differentiate both sides of the equation with respect to x .
2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation.
3. Factor out $\frac{dy}{dx}$.
4. Solve for $\frac{dy}{dx}$.

Lenses, Tangents and Normal Lines

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Figure 3.50). This line is called the *normal to the surface* at the point of entry. In a profile view of a lens, the normal is a line perpendicular to the tangent to the profile curve at the point of entry.

Implicit differentiation is often used to find the tangents and normals of lenses described as quadratic curves.

Lenses, Tangents and Normal Lines



Example Lenses, Tangents and Normal Lines

Find the equations of the tangent and normal lines to the graph

given by $x^4 + x^2 y^2 - y^2 = 0$ at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$.

Differentiate implicitly as

$$\frac{d}{dx}(x^4) + \frac{d}{dx}(x^2 y^2) - \frac{d}{dx}(y^2) = 0$$

$$4x^3 + x^2 \left(2y \frac{dy}{dx} \right) + 2xy^2 - 2y \frac{dy}{dx} = 0 \quad \text{treat } x^2 y^2 \text{ as a product}$$

$$2y(x^2 - 1) \frac{dy}{dx} = -2x(2x^2 + y^2)$$

$$\frac{dy}{dx} = \frac{-2x(2x^2 + y^2)}{2y(x^2 - 1)}$$

$$\frac{dy}{dx} = - \frac{x(2x^2 + y^2)}{y(x^2 - 1)} = \text{slope of the tangent line}$$

Example Lenses, Tangents and Normal Lines

$$\frac{dy}{dx} = -\frac{x(2x^2 + y^2)}{y(x^2 - 1)} \quad \text{Find the slope of the tangent line at } \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)$$

$$m = -\frac{\frac{\sqrt{2}}{2} \left(2 \left(\frac{\sqrt{2}}{2} \right)^2 + \left(\frac{\sqrt{2}}{2} \right)^2 \right)}{\frac{\sqrt{2}}{2} \left(\left(\frac{\sqrt{2}}{2} \right)^2 - 1 \right)} = -\frac{\frac{\sqrt{2}}{2} \left(2 \left(\frac{2}{4} \right) + \left(\frac{2}{4} \right) \right)}{\frac{\sqrt{2}}{2} \left(\left(\frac{2}{4} \right) - 1 \right)} = -\frac{\frac{3}{2}}{-\frac{1}{2}} = 3$$

The equation of the **tangent line** at this point is

$$y - \frac{\sqrt{2}}{2} = 3 \left(x - \frac{\sqrt{2}}{2} \right) \Rightarrow y = 3x - \sqrt{2}.$$

The equation of the **normal line** at this point is

$$y - \frac{\sqrt{2}}{2} = -\frac{1}{3} \left(x - \frac{\sqrt{2}}{2} \right) \Rightarrow y = -\frac{1}{3}x + \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{2}$$

$$y = -\frac{1}{3}x + \frac{2\sqrt{2}}{3}$$

Example Derivatives of a Higher Order

Find $\frac{d^2x}{dy^2}$ if $x^2 + y^2 = 25$

Differentiating each term with respect to x gives

$$2x + 2y \frac{dy}{dx} = 0$$

$$2y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Differentiating again with respect to x gives

$$\frac{d^2x}{dy^2} = -\frac{(y)(1) - (x)\left(\frac{dy}{dx}\right)}{y^2}$$

Quotient Rule

$$= -\frac{y - (x)\left(-\frac{x}{y}\right)}{y^2}$$

Substitute $-\frac{x}{y}$ for $\frac{dy}{dx}$

$$= -\frac{y^2 + x^2}{y^3}$$

Simplify

Rule 9

Power Rule For Rational Powers of x

If n is any rational number, then

$$\frac{d}{dx} x^n = nx^{n-1}.$$

If $n < 1$, then the derivative does not exist at $x=0$.

Quick Quiz Sections 4.1 – 4.2

You should solve the following problems without using a graphing calculator.

1. Which of the following gives $\frac{dy}{dx}$ for $y = \sin^4(3x)$?

(A) $4\sin^3(3x)\cos(3x)$

(B) $12\sin^3(3x)\cos(3x)$

(C) $12\sin(3x)\cos(3x)$

(D) $12\sin^3(3x)$

(E) $-12\sin^3(3x)\cos(3x)$

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(D) $12\sin^3(3x)$

(E) $-12\sin^3(3x) \cos(3x)$

Quick Quiz Sections 4.1 – 4.2

2. Which of the following gives y'' for $y = \cos x + \tan x$?

(A) $-\cos x + 2\sec^2 x \tan x$

(B) $\cos x + 2\sec^2 x \tan x$

(C) $-\sin x + \sec^2 x$

(D) $-\cos x + \sec^2 x \tan x$

(E) $\cos x + \sec^2 x \tan x$

Quick Quiz Sections 4.1 – 4.2

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(D) $-\cos x + \sec^2 x \tan x$

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Quick Quiz Sections 4.1 – 4.2

3. Which of the following gives $\frac{dy}{dx}$ for the parametric curve

$$x=3\sin t, y=2\cos t?$$

(A) $-\frac{3}{2}\cot t$

(B) $\frac{3}{2}\cot t$

(C) $-\frac{2}{3}\tan t$

(D) $\frac{2}{3}\tan t$

(E) $\tan t$

Quick Quiz Sections 4.1 – 4.2

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(C) $-\frac{2}{3}\tan t$

(D) $\frac{2}{3}\tan t$

(E) $\tan t$